3329. Proposed by Arkady Alt, San Jose, CA, USA.

Let r be a real number, $0 < r \le 1$, and let x, y, and z be positive real numbers such that $xyz = r^3$. Prove that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \le \frac{3}{\sqrt{1+r^2}}.$$

Solution.

First we will prove inequality

(1) $\quad \dfrac{1}{\sqrt{1+x^2}} + \dfrac{1}{\sqrt{1+y^2}} \le \dfrac{2}{\sqrt{1+r^2}}$ for $x, y > 0$ such that $xy = r^2, r \le 1$.

$$\sqrt{1+r^2}\left(\frac{2}{\sqrt{1+r^2}} - \frac{1}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+y^2}}\right) = \frac{\sqrt{1+x^2} - \sqrt{1+r^2}}{\sqrt{1+x^2}} +$$

$$\frac{\sqrt{1+y^2} - \sqrt{1+r^2}}{\sqrt{1+y^2}} = \frac{x(x-y)}{1+x^2+\sqrt{1+x^2}\sqrt{1+r^2}} + \frac{y(y-x)}{1+y^2+\sqrt{1+r^2}\sqrt{1+y^2}} =$$

$$\frac{(y-x)\left(y\left(1+x^2+\sqrt{1+x^2}\sqrt{1+r^2}\right) - x\left(1+y^2+\sqrt{1+r^2}\sqrt{1+y^2}\right)\right)}{\left(1+x^2+\sqrt{1+x^2}\sqrt{1+r^2}\right)\left(1+y^2+\sqrt{1+r^2}\sqrt{1+y^2}\right)} =$$

$$\frac{(y-x)^2\left(1-r^2+\dfrac{\sqrt{1+r^2}\,(y+x)}{\left(y\sqrt{1+x^2}+x\sqrt{1+y^2}\right)}\right)}{\left(1+x^2+\sqrt{1+x^2}\sqrt{1+r^2}\right)\left(1+y^2+\sqrt{1+r^2}\sqrt{1+y^2}\right)} \ge 0$$ since

$$y\left(1+x^2+\sqrt{1+x^2}\sqrt{1+r^2}\right) - x\left(1+y^2+\sqrt{1+r^2}\sqrt{1+y^2}\right) =$$

$$y - x - xy(y-x) + y\sqrt{1+x^2}\sqrt{1+r^2} - x\sqrt{1+r^2}\sqrt{1+y^2} =$$

$$y - x - r^2(y-x) + \frac{\sqrt{1+r^2}\,(y^2-x^2)}{\left(y\sqrt{1+x^2}+x\sqrt{1+y^2}\right)} =$$

$$(y-x)\left(1-r^2+\frac{\sqrt{1+r^2}\,(y+x)}{\left(y\sqrt{1+x^2}+x\sqrt{1+y^2}\right)}\right)$$ and $r \le 1$.

Now, using inequality (**1**) in the supposition that real positive $r \le 1$ we will prove that

(**2**) $\quad \dfrac{1}{\sqrt{1+x^2}} + \dfrac{1}{\sqrt{1+y^2}} + \dfrac{1}{\sqrt{1+z^2}} \le \dfrac{3}{\sqrt{1+r^2}}$ for any $x, y, z > 0$ such that $xyz = r^3$.

Due to the symmetry of inequality (2) we can suppose that $x \le y \le z$.

Denote $t := \sqrt{xy}$, then $z := \dfrac{r^3}{t^2}$, $t \le z \Leftrightarrow t \le r$ and since $t \le 1$ we can apply inequality (1):

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \le \frac{2}{\sqrt{1+t^2}} + \frac{1}{\sqrt{1+z^2}}.$$

Let $h(t) = \dfrac{2}{\sqrt{1+t^2}} + \dfrac{1}{\sqrt{1+z^2}}$, then $z' = -\dfrac{2r^3}{t^3} = -\dfrac{2z}{t}$ and

$$h'(t) = -\frac{2t}{\sqrt{(1+t^2)^3}} - \frac{zz'}{\sqrt{(1+z^2)^3}} = -\frac{2t}{\sqrt{(1+t^2)^3}} + \frac{2z^2}{t\sqrt{(1+z^2)^3}} =$$

$$\frac{2z^2\left(z^2\sqrt{(1+t^2)^3}-t^2\sqrt{(1+z^2)^3}\right)}{t\sqrt{(1+t^2)^3(1+z^2)^3}}=\frac{2z^2\left(z^4(1+t^2)^3-t^4(1+z^2)^3\right)}{t\sqrt{(1+t^2)^3(1+z^2)^3}\left(z^2\sqrt{(1+t^2)^3}+t^2\sqrt{(1+z^2)^3}\right)}.$$

Note that $z^4(1+t^2)^3-t^4(1+z^2)^3=(z^2-t^2)(z^2+t^2+3z^2t^2-z^4t^4)\geq 0,$ since $z\geq t$ and $z^2+t^2+3z^2t^2-z^2(zt^2)^2=z^2+t^2+3z^2t^2-z^2r^6=z^2(1-r^6)+t^2+3z^2t^2>0.$

Thus, $h'(t)>0$ for $t<r$ and $h'(r)=0$ we conclude that $h(t)$ increasing function on $(0,r]$ and $\max_{t\in(0,r]}h(t)=h(r)=\frac{3}{\sqrt{1+r^2}}$.